

Equality of Dedekind sums modulo $24\mathbb{Z}$

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Abstract

Let $S(a, b) = 12s(a, b)$, where $s(a, b)$ denotes the classical Dedekind sum. In a recent note E. Tsukerman gave a necessary and sufficient condition for $S(a_1, b) - S(a_2, b) \in 8\mathbb{Z}$. In the present paper we show that this condition is equivalent to $S(a_1, b) - S(a_2, b) \in 24\mathbb{Z}$, provided that $9 \nmid b$. Tsukerman also obtained a congruence mod 8 for $bT(a, b)$, where $T(a, b)$ is the alternating sum of the partial quotients of the continued fraction expansion of a/b . We show that the respective congruence holds mod 24 if $3 \nmid b$ and mod 72 if $3 \mid b$.

1. Introduction and results

Let a be an integer, b a natural number, and $(a, b) = 1$. The classical Dedekind sum $s(a, b)$ is defined by

$$s(a, b) = \sum_{k=1}^b ((k/b))((ak/b)).$$

Here

$$((x)) = \begin{cases} x - [x] - 1/2 & \text{if } x \in \mathbb{R} \setminus \mathbb{Z}; \\ 0 & \text{if } x \in \mathbb{Z} \end{cases}$$

(see [2, p. 1]). It is often more convenient to work with

$$S(a, b) = 12s(a, b)$$

instead (see, for instance, formula (7) below). Since $S(a + b, b) = S(a, b)$, we obtain all Dedekind sums if a is restricted to the range $0 \leq a \leq b - 1$, $(a, b) = 1$.

In the recent note [4], E. Tsukerman gave a necessary and sufficient condition for the equality of $S(a_1, b)$ and $S(a_2, b)$ modulo $8\mathbb{Z}$. This condition involves the function μ , which is defined, for a, b as above, as follows

$$\mu(a, b) = \begin{cases} 2 - 2\left(\frac{a}{b}\right), & \text{if } b \text{ is odd;} \\ (a-1)(a+b-1), & \text{if } b \text{ is even.} \end{cases}$$

Here $\left(\frac{a}{b}\right)$ is the Jacobi symbol. Tsukerman's condition is phrased by means of the residue class

$$b(a_2\mu(b, a_1) - a_1\mu(b, a_2)) \pmod{8b}.$$

We observe, however, that this residue class depends only on the residue classes of $\mu(b, a_1)$ and $\mu(b, a_2)$ modulo 8, not of the values of $\mu(b, a_1)$ and $\mu(b, a_2)$ themselves. Therefore,

we may replace the function μ by the following simpler function, which we henceforth also call μ .

$$\mu(a, b) = \begin{cases} 2 - 2\left(\frac{a}{b}\right), & \text{if } b \text{ is odd;} \\ 4, & \text{if } b \equiv 0 \pmod{4} \text{ and } a \equiv 3 \pmod{4}; \\ 0, & \text{otherwise.} \end{cases}$$

In this paper we show

Theorem 1 *Let $a_1, a_2 \in \mathbb{N}$ be relatively prime to $b \in \mathbb{N}$. Suppose, further, that $9 \nmid b$. Then*

$$S(a_1, b) - S(a_2, b) \in 24\mathbb{Z}$$

if, and only if,

$$b(a_2\mu(b, a_1) - a_1\mu(b, a_2)) \equiv (a_1 - a_2)(b - 1)(a_1a_2 + b - 1) \pmod{8b}. \quad (1)$$

This equivalence cannot be extended to the case $9 \mid b$ in an obvious way, as we show in Section 3. Tsukerman showed that (1) is equivalent to $S(a_1, b) - S(a_2, b) \in 8\mathbb{Z}$ for arbitrary natural numbers b , i.e., he needed not assume $9 \nmid b$.

For $a \in \mathbb{Z}$ and $b \in \mathbb{N}$, let

$$\frac{a}{b} = [a_0, a_1, \dots, a_n]$$

denote the regular continued fraction expansion of a/b . The partial quotients a_1, \dots, a_n are natural numbers. We do not assume $a_n \geq 2$, but require n to be odd, instead. Define

$$T(a, b) = \sum_{k=0}^n (-1)^{k-1} a_k. \quad (2)$$

In the said paper, Tsukerman showed, for $a, b \in \mathbb{N}$, $(a, b) = 1$,

$$bT(a, b) \equiv -\mu(a, b) + b^2 + 2 - a - a^* \pmod{8}, \quad (3)$$

with $a^* \in \{1, \dots, b-1\}$, $aa^* \equiv 1 \pmod{b}$. Our new definition of μ suggests a more explicit form of (3), which we use in the following Theorem. To this end we define $\varepsilon \in \{\pm 1\}$ by the congruence

$$a \equiv \varepsilon \pmod{3} \quad (4)$$

for each $a \in \mathbb{Z}$, $3 \nmid a$.

Theorem 2 *Let $a \in \mathbb{Z}$ be relatively prime to $b \in \mathbb{N}$.*

(a) Let b be odd. If $3 \nmid b$, then

$$bT(a, b) \equiv 9 + 18\left(\frac{a}{b}\right) - a - a^* \pmod{24}.$$

If $3 \mid b$, then

$$bT(a, b) \equiv 9 + 18\left(\frac{a}{b}\right) - 16\varepsilon - a - a^* \pmod{72}.$$

(b) Let $b \equiv 2 \pmod{4}$ or let both $b \equiv 0 \pmod{4}$ and $a \equiv 3 \pmod{4}$ hold. If $3 \nmid b$, then

$$bT(a, b) \equiv 6 - a - a^* \pmod{24}.$$

If $3 \mid b$, then

$$bT(a, b) \equiv 54 - 16\varepsilon - a - a^* \pmod{72}.$$

(c) Let $b \equiv 0 \pmod{4}$ and $a \equiv 1 \pmod{4}$. If $3 \nmid b$, then

$$bT(a, b) \equiv 18 - a - a^* \pmod{24}.$$

If $3 \mid b$, then

$$bT(a, b) \equiv 18 - 16\varepsilon - a - a^* \pmod{72}.$$

In Section 3 we exhibit many examples that illustrate both Theorem 1 and the fact that this theorem does not hold if $9 \mid b$.

2. Proofs

Our main tools are two congruences modulo 3 for Dedekind sums. First we observe that $bS(a, b)$ is an integer; moreover, if 3 does not divide b , then

$$bS(a, b) \equiv 0 \pmod{3}. \quad (5)$$

These assertions follow from [2, p. 27, Th. 2]). On the other hand, if $3 \mid b$,

$$bS(a, b) \equiv 2\varepsilon \pmod{9}, \quad (6)$$

where ε is defined as in (4) (see [3, formula (70)]).

Proof of Theorem 1. Suppose, first, that $3 \nmid b$. Because of (5), we may write

$$S(a_1, b) = \frac{3k_1}{b}, \quad S(a_2, b) = \frac{3k_2}{b}$$

with integers k_1, k_2 . By [4, Th. 3.1], the congruence (1) is equivalent to $S(a_1, b) - S(a_2, b) \in 8\mathbb{Z}$. Accordingly, (1) is also equivalent to

$$\frac{3(k_1 - k_2)}{b} = 8r, \quad r \in \mathbb{Z}.$$

However, $3 \nmid b$, and so this means $3 \mid r$. This proves Theorem 1 in the case $3 \nmid b$. Suppose now that $3 \mid b$. Then the congruence (1) implies $(a_1 - a_2)(a_1 a_2 - 1) \equiv 0 \pmod{3}$. Hence we obtain, from (6)

$$S(a_1, b) = \frac{2\varepsilon + 9k_1}{b}, \quad S(a_2, b) = \frac{2\varepsilon + 9k_2}{b}$$

with a common value $\varepsilon \equiv a_1 \equiv a_2 \pmod{3}$ and $k_1, k_2 \in \mathbb{Z}$. Accordingly, (1) is equivalent to

$$\frac{9(k_1 - k_2)}{b} = 8r, \quad r \in \mathbb{Z}.$$

If $9 \mid b$, this simply means $S(a_1, b) - S(a_2, b) \in 8\mathbb{Z}$, so this is just Tsukerman's result. However, if $9 \nmid b$, we obtain $3 \mid r$, which yields the theorem in the case $3 \mid b, 9 \nmid b$. \square

Proof of Theorem 2. The Barkan-Hickerson-Knuth formula says

$$S(a, b) = T(a, b) + \frac{a + a^*}{b} - 3 \quad (7)$$

(see, for instance, [1]). Note that this formula is often enunciated only for the case $0 \leq a < b$, but it is, in fact, valid for arbitrary integers a relatively prime to b , provided that $T(a, b)$ is defined as in (2). Hence we obtain, by (5) and (7),

$$bT(a, b) \equiv -a - a^* \pmod{3} \quad (8)$$

if $3 \nmid b$. In the case $3 \mid b$, (6) and (7) give

$$bT(a, b) \equiv 2\varepsilon - a - a^* \pmod{9} \quad (9)$$

instead. Further, Tsukerman's congruence (3) is also valid for arbitrary integers a relatively prime to b , as we easily check. We combine (3) with the congruences (8), (9) by means of the Chinese remainder theorem. This readily gives Theorem 2. \square

3. A proposition yielding examples

Our examples arise from the following proposition.

Proposition 1 *Let c, d be odd natural numbers, $d \geq 3$. Put $b = cd^2$ and $a = cd + 1$. Then*

$$S(1, b) - S(a, b) = c(d^2 - 1).$$

Proof. We apply the reciprocity law for Dedekind sums (see [2, p. 5]), which gives

$$S(a, b) = -S(b, a) + \frac{a}{b} + \frac{b}{a} + \frac{1}{ab} - 3.$$

Now $b \equiv -d \pmod{a}$, hence the reciprocity law says

$$S(b, a) = S(-d, a) = S(a, d) - \frac{a}{d} - \frac{d}{a} - \frac{1}{ad} + 3.$$

However, $a \equiv 1 \pmod{d}$, and so $S(a, d) = S(1, d) = d - 3 + 2/d$. Inserting the values $b = cd^2$ and $a = cd + 1$ gives

$$S(a, b) = c - 3 + \frac{2}{b}.$$

Since $S(1, b) = b - 3 + 2/b$, we obtain the desired result. \square

In the setting of the proposition, let $3 \nmid d$. Then $d^2 - 1 \equiv 0 \pmod{24}$, so the proposition yields many examples with $S(1, b) - S(a, b) > 0$ and $S(1, b) - S(a, b) \equiv 0 \pmod{24}$. On the other hand, if $3 \mid d$, then $d^2 - 1 \equiv 0 \pmod{8}$, but $d^2 - 1 \not\equiv 0 \pmod{24}$. If, therefore, $3 \nmid c$, we obtain many examples with $S(1, b) - S(a, b) \equiv 0 \pmod{8}$, but $S(1, b) - S(a, b) \not\equiv 0 \pmod{24}$.

References

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